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# Transition between stimulated backscattering and soliton exchange in ferrites

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**Abstract.** We study the interaction of three monochromatic polarized electromagnetic waves in a ferromagnetic dielectric, and show that it is governed by the three-wave resonant interaction system, a completely integrable system. We solve the resonance condition and classify the cases where the interaction occurs; for high frequencies, we characterize the solutions completely. Thus we find transition points between the regimes where stimulated backscattering of one of the waves or soliton exchange occurs.

## 1. Introduction

The propagation of electromagnetic waves in ferromagnetic media was studied in the late 1950s, in relation to ferrite devices at microwaves frequencies. Highly nonlinear effects were observed. Among them, we are here especially interested in the resonant interaction of three waves. As early as 1958, Tien and Suhl [1] described such an interaction in a ferrite, in order to apply this effect to build an amplifier. Recently, various studies of this type of interaction, still in ferrites, have been published. They mainly concern layer materials [2, 3], and show the generation of uniform magnetic precession and of waves with opposite directions. Apart from a discussion, based on experimental data, of the possibility of observing these effects, these works derive theoretically the wave equations that describe the interaction; that is the so-called three-wave resonant interaction (3WRI) system, which is somehow universal, in the sense that it occurs in various physical domains (hydrodynamics, plasma physics, and so on; a review of the abundant literature that exists on this subject can be found, e.g., in [4]). Important theoretical work has been done on this 'universal' model. It has been shown, using the multiscale expansion method, that it can be derived as an asymptotic mode from very general evolution equations allowing wave propagation [5, 6]. Furthermore, it is completely integrable by the inverse scattering transform (IST) method, and the properties of its solutions have been studied in detail (see [4], and the literature quoted in it).

The aim of the present article is to study the interaction between three electromagnetic waves in a ferro- (or ferri-)magnetic medium, by a rigorous use of the multiscale expansion method, and application of the results of soliton theory. The evolution of the electromagnetic field $\boldsymbol{H}$ and the magnetization density $\boldsymbol{M}$ in such a medium is governed by the equations:

$$-\nabla(\nabla\cdot\boldsymbol{H})+\Delta\boldsymbol{H}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\boldsymbol{H}+\boldsymbol{M}) \tag{1}$$

$$\frac{\partial\boldsymbol{M}}{\partial t}=-\mu_0\delta\boldsymbol{M}\wedge\boldsymbol{H} \tag{2}$$

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where $\mu_0$ is the magnetic permeability in vacuum, $\delta$ the gyromagnetic ratio and $c = 1/\sqrt{\hat{\epsilon}\mu_0}$ is the speed of light based on the electric permittivity $\hat{\epsilon}$ of the medium; all formulae below will be given in terms of the rescaled variables defined in appendix A (A1).

Equation (1) follows directly from the Maxwell equations in a dieletric medium with a linear constitutive relation for the electric part; equation (2) describes the evolution of the magnetization density in a ferro- or ferri-magnetic medium; this model is the basis of most studies of electromagnetic wave propagation in ferrites [7–9], in particular ferromagnetic resonance. The use of this model needs some further assumptions: our medium is assumed to be infinite and isotropic, and we neglect the microscopic effects of inhomogeneous exchange interaction and of damping. Furthermore, the sample is assumed to be immersed in an external constant magnetic field, and saturated.

In a previous paper [10], it was shown that taking account of the damping in this system, the evolution of an electromagnetic perturbation of long wavelength is governed by Burgers' equation, which provided solutions that describe the propagation and the coalescence of travelling waves of determined phase velocity in $(1 + 1)$ and $(2 + 1)$ dimensions.

In other work [7, 11] the nonlinear modulation of amplitude of a monochromatic wave in a ferrite was studied. There the damping was neglected and the system reduced to the nonlinear Schrödinger equation (NLS). As a result we showed the existence of a Benjamin–Feir instability and characterized it in terms of the physical parameters of the system.

In this work, we deal again with monochromatic waves in the same medium, but instead of looking at the nonlinear self-interaction of the wave, we will consider the interaction of three waves. The idea of studying such a problem follows naturally from the observation that the dispersion relation of such waves contains three branches, two of which may be called 'optic', and the other 'acoustic'.

Equations (1) and (2) are reduced by the perturbative method of stretched coordinates to the 3WRI system, which is integrable by means of the IST method. This is possible only if some resonance condition is satisfied. We solve it in the general case by a graphic method, and give asymptotic values for the solutions for high frequencies. The behaviour of the 3WRI is characterized by the sign of the interaction coefficients and the relative magnitude of the velocities. We determine these quantities in terms of the physical parameters and discuss the type of interaction encountered. Among the three types of interaction that may be described by the 3WRI system, two occur here: one is called stimulated backscattering (SBS). This is the more general case of stimulated Brillouin scattering, where the acoustic wave is replaced by an electromagnetic wave: the wave with the highest velocity (an 'optical' wave in most cases) is reflected by a wave of intermediate velocity (an 'acoustic' wave in most cases).

The other one is called the soliton exchange interaction, and its most important effect is that any soliton contained in the wave with intermediate velocity disappears, giving rise to two solitons, one in each of the other waves. This transfer appears as soon as the slightest amount of one of the two waves with extreme velocities is present in the interaction region (random noise is sufficient).

The cases where 3WRI occurs, i.e. the solutions of the resonance condition obtained through our graphic method, can be grouped into three classes. For two of them, all the waves may have large wavenumbers and the asymptotic solution can be derived. In this case the interaction is shown to be of SBS type, with or without change of polarization, depending on the class. For the third class, only two among the three frequencies are allowed to take arbitrary large values, and the third then has a finite limit. Therefore, the solution cannot be performed as completely as previously; we give results for the two particular cases where the propagation is parallel to the external field, and where the magnitude of the field is large.

Transition points between SBS and soliton exchange interaction appear in several cases; these transitions occur as the magnitude of the external field passes through some critical values, which are algebraically and numerically calculated, but also as the angle between the external field and the propagation direction crosses critical values too.

## 2. The formalism

Consider first the dispersion relation of the system (1), (2), for a wave of pulsation $\omega$ and wavenumber $k$, propagating in the $x$ direction, in an external magnetic field $\boldsymbol{H}^0 = \alpha \boldsymbol{m}$ such that the magnetization density is $\boldsymbol{m} = (m_x, m_t, 0)$. It reads:

$$\left(1+\alpha\left(1-\frac{k^2}{\omega^2}\right)\right)^2 m_x^2 + \left(1-\frac{k^2}{\omega^2}\right)\left(1+\alpha\left(1-\frac{k^2}{\omega^2}\right)\right)(1+\alpha)m_t^2 = \left(1-\frac{k^2}{\omega^2}\right)^2 \omega^2. \tag{3}$$

The solutions $\omega(k)$ of this equation are plotted in figure 1. The figure presents three branches; for each of them the polarization is entirely defined. There are two branches with positive helicity, which we call PO and PA, and one with negative helicity, N. Owing to the fact that as $k \longrightarrow 0$, $\omega$ tends to $\omega_0 = (1+\alpha)m$, $(m = \|\boldsymbol{m}\|)$ on the PO branch, it may be called the optic one, and, conversely, branch PA may be called acoustic. The limit of $\omega$ on the latter as $k \longrightarrow +\infty$ is $\omega_l = m\sqrt{\alpha(\alpha + \sin^2\varphi)}$ ($\varphi$ is such that $m_x = m\cos\varphi$, $m_t = m\sin\varphi$). The third branch, N, behaves like the acoustic positive one for small values of $k$ and like the optic positive one for large values of $k$.

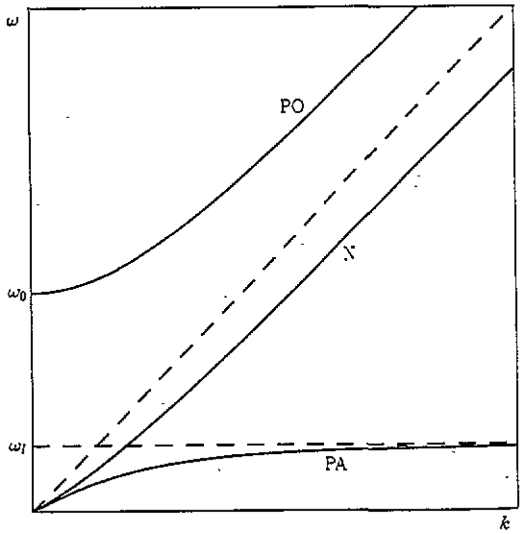


**Figure 1.** Dispersion relation $\omega(k)$ of an electromagnetic wave in ferrite. P (N) refers to the positive (negative) helicity of the wave.

The resonance condition of our interaction problem reads as follows: if $(k_j, \omega_j)_{j=1,2,3}$ are the wavenumbers and pulsations of the three waves, we must have (Brillouin selection rules):

$$k_1 = k_2 + k_3 \qquad \omega_1 = \omega_2 + \omega_3. \tag{4}$$

In section 3 we will see how this condition can be realized with dispersion relation (3).

Both conditions (3) and (4) are obtained using a perturbative method, that leads to the reduction of the basic equations to the integrable 3WRI system. This method works as follows. Let

$$\begin{aligned} \phi_j &= k_j x - \omega_j t \\ \phi_{-j} &= -\phi_j \end{aligned} \qquad j = 1, 2, 3. \tag{5}$$

We expand $\boldsymbol{H}$ in series of powers of the $\mathrm{e}^{\mathrm{i}\phi_j}$'s and of a small parameter $\varepsilon$ in the following way:

$$\boldsymbol{H} = \boldsymbol{H}^0 + \varepsilon \sum_j \boldsymbol{H}_j^1 \mathrm{e}^{\mathrm{i}\phi_j} + \varepsilon^2 \Bigg( \sum_{n\in\mathbb{Z}^3} \boldsymbol{H}_n^2 \mathrm{e}^{\mathrm{i}n\cdot\phi} \Bigg) + \mathrm{O}(\varepsilon^3) \tag{6}$$

where

$$n \cdot \phi = n_1\phi_1 + n_2\phi_2 + n_3\phi_3.$$

$\boldsymbol{H}_0$ is the constant exterior field in which the sample is immersed, and each of the quantities $\boldsymbol{H}_j^1 \mathrm{e}^{\mathrm{i}\phi_j} + \mathrm{CC}$, $j = 1, 2, 3$, represents a monochromatic wave propagating with wavenumber $k_j$ and pulsation $\omega_j$ in the $x$ direction. Expansion (6) thus represents three monochromatic waves of the same order of magnitude, immersed in a constant exterior field. The small quantity $\varepsilon$ is the ratio between the wave field and the exterior field. We expand the magnetization $\boldsymbol{M}$ in an analogous way.

The amplitudes $\boldsymbol{H}_j^1$ of the three waves under consideration (and also the amplitudes $\boldsymbol{H}_n^2$ of the second harmonics) are assumed to vary slowly in time and space. Thus, according to the multiple scale expansion method, these quantities will depend on stretched coordinates $\xi$ and $\tau$ defined by

$$\xi = \varepsilon x \qquad \tau = \varepsilon t. \tag{7}$$

In order to study an interaction between waves with different group velocities, we must choose the same order of magnitude for both time and space variables: every other choice selects a wave with a fixed group velocity. Furthermore, the scaling we choose means that the ratio between the typical length of the amplitude modulation and the wavelength is large, having an order of magnitude $1/\varepsilon$, the inverse of the ratio between the wave field and the exterior field. Making this choice, the first nonlinear term appears in the equations describing the interaction of the three waves. For a lower intensity of the incident waves, the nonlinear variation in the amplitudes occurs on a larger space scale (thus no interaction appears for the chosen scale). Conversely, if the intensity increases, the weakly nonlinear approximation that we intend to study here occurs at a smaller space scale (thus at the scale under consideration, we should have to take into account the higher-order nonlinear interaction terms).

We put these expansions into the basic equations (1), (2) and, collecting the terms order by order, we obtain (more detail on the derivation of these results is given in appendix A):

(i) At order $\varepsilon^0$: $\boldsymbol{H}^0$ must be collinear to $\boldsymbol{M}^0$. Let $\alpha$ be such that $\boldsymbol{H}^0 = \alpha \boldsymbol{M}^0$, we choose the axes such that

$$\boldsymbol{M}^0 = \boldsymbol{m} = (m_x, m_t, 0). \tag{8}$$

(ii) At order $\varepsilon^1$: for each $j = 1, 2, 3$ we obtain

$$\begin{aligned} \boldsymbol{M}_j^1 &= \boldsymbol{m}_j^1 g_j \\ \boldsymbol{H}_j^1 &= \boldsymbol{h}_j^1 g_j \end{aligned} \tag{9}$$

where $g_j$ is an unknown function of $(\xi, \tau)$, and $\boldsymbol{m}_j^1$, $\boldsymbol{h}_j^1$ are polarization vectors, functions of $\boldsymbol{m}$, $\alpha$, $\omega_j$, $k_j$.

(iii) At order $\varepsilon^2$, we see first that resonance condition (4) allows interaction terms to appear, equating terms proportional to $\varepsilon^2 e^{i\phi_1}$ with those that come from the product of $\varepsilon M_2^1 e^{i\phi_2}$ by $\varepsilon M_3^1 e^{i\phi_3}$, and so on.

Writing the condition for the existence of the $M_j^2$ terms, we obtain that $g_1$, $g_2$, $g_3$ obey the 3WRI system:

$$\begin{aligned}
&\frac{\partial}{\partial\tau}g_1 + V_1\frac{\partial}{\partial\xi}g_1 = A_1 g_2 g_3\\
&\frac{\partial}{\partial\tau}g_2 + V_2\frac{\partial}{\partial\xi}g_2 = A_2 g_1 g_3^*\\
&\frac{\partial}{\partial\tau}g_3 + V_3\frac{\partial}{\partial\xi}g_3 = A_3 g_1 g_2^*
\end{aligned}\tag{10}$$

where the $V_j$ are the group velocities of the waves:

$$V_j = \frac{b_j + 1}{b_j + 1 + \gamma_j\mu_j u_j^2}u_j \tag{11}$$

and the $A_j$ the interaction constants (real):

$$A_j = -\frac{\mu_j u_j^2}{2\Gamma_j\omega_j}\frac{A_j'}{(b_j + 1 + \gamma_j\mu_j u_j^2)} \tag{12}$$

$$\begin{aligned}
A_1' = \omega_1 m_x m_t\{&\gamma_1\omega_1\mu_2\mu_3[\gamma_2(1-\gamma_3) + \gamma_3(1-\gamma_2)]\\
&+\mu_1\gamma_2\gamma_3[\mu_2\omega_3(1-\gamma_3) + \mu_3\omega_2(1-\gamma_2)]\\
&-\gamma_1(1+\alpha)(\gamma_3-\gamma_2)[\mu_3\gamma_2\omega_2 - \mu_2\gamma_3\omega_3]\}
\end{aligned}\tag{13}$$

and analogous expressions for $A_2'$, $A_3'$, which are given in appendix A. In these formulae we have put

$$u_j = \omega_j/k_j \tag{14a}$$

$$\gamma_j = 1 - 1/u_j^2 \qquad \mu_j = 1 + \alpha\gamma_j \tag{14b}$$

$$\Gamma_j = \gamma_j^2\omega_j^2 \qquad b_j = \mu_j^2 m_x^2/\gamma_j^2\omega_j^2. \tag{14c}$$

System (10) can be reduced to

$$\begin{aligned}
&\frac{\partial}{\partial\tau}q_1 + V_1\frac{\partial}{\partial\xi}q_1 = \delta_1 q_2^* q_3^*\\
&\frac{\partial}{\partial\tau}q_2 + V_2\frac{\partial}{\partial\xi}q_2 = \delta_2 q_1^* q_3^*\\
&\frac{\partial}{\partial\tau}q_3 + V_3\frac{\partial}{\partial\xi}q_3 = \delta_3 q_1^* q_2^*
\end{aligned}\tag{15}$$

where the $\delta_j$ are the signs of the $A_j$, by putting

$$g_1 = \frac{q_1}{\sqrt{|A_2A_3|}} \qquad g_2 = \frac{q_2^*}{\sqrt{|A_1A_3|}} \qquad g_3 = \frac{q_3^*}{\sqrt{|A_1A_2|}}. \tag{16}$$

The system (15) (or (10)) is completely integrable by the IST method, and has been studied by many authors, see e.g. [12, 4].

In section 3 of this article, we will give a graphic solution of the resonance condition (4); in sections 4 and 5, we will study extensively the limiting case of high frequencies, for which asymptotic developments can be given for the solution of the resonance condition, the group velocities and the interaction coefficients. Then, in each case, we will give a physical interpretation of the results, using the known solution of (15).

## 3. Graphic resolution of the resonance condition

The wavenumbers $k_1$, $k_2$, $k_3$ and pulsations $\omega_1$, $\omega_2$, $\omega_3$ of the three waves must satisfy the resonance condition (4):

$$k_1 = k_2 + k_3 \qquad \omega_1 = \omega_2 + \omega_3$$

and the dispersion relation (3), which can be written

$$\mu_j^2 m_x^2 + \gamma_j \mu_j (1+\alpha) m_t^2 = \gamma_j^2 \omega_j^2 \tag{17}$$

where $\gamma_j$, $\mu_j$ are given by (14*b*), and $\alpha$, $m = (m_x, m_t, 0)$ by (8).

The complexity of (17) (the degree in $\omega$ is 6) prevents any analytic solution of the whole problem. We can take the $\omega_j$ positive without loss of generality, but the $k_j$ may be positive or negative, corresponding to a propagation towards the $x$ positive or in the opposite direction. We can only fix the sign of one of them, let $k_1 > 0$. Let us call $M_j$ the points of coordinates $(k_j, \omega_j)$ in the $(k, \omega)$ plane, O being the origin. Condition (4) means that $OM_2M_1M_3$ is a parallelogram, i.e. that $M_2$ and $M_3$ are symmetric with respect to the mid-point A of $[OM_1]$. Thus, for a given $M_1$, you have only to construct the curves (PA′, N′, PO′) symmetric to the graph (PA, N, PO) of the dispersion relation, with respect to A; the intersection of the branches (PA′, N′, PO′) and (PA, N, PO) gives the points that we seek.

There are three cases to be considered, according to which of the three branches PA, N and PO contains $M_1$. A permutation between $M_2$ and $M_3$ has no significant effect so we will often assume that $\omega_2 > \omega_3$.

(i) $M_1 \in$ PO. Figure 2 gives the construction. We see that there are two solutions (1a), (1b) with $M_2$ on N, with opposite directions of propagation, and one (1c) with $M_2$ on the same branch PO as $M_1$, but in the opposite direction. In the three cases $M_3$ is on the PA branch. In the fourth solution (1d), $M_2$ and $M_3$ are on the N branch, in opposite directions. In the fifth (1e), $M_2$ is on the N branch and $M_3$ in the same branch OP as $M_1$, but in the opposite direction.

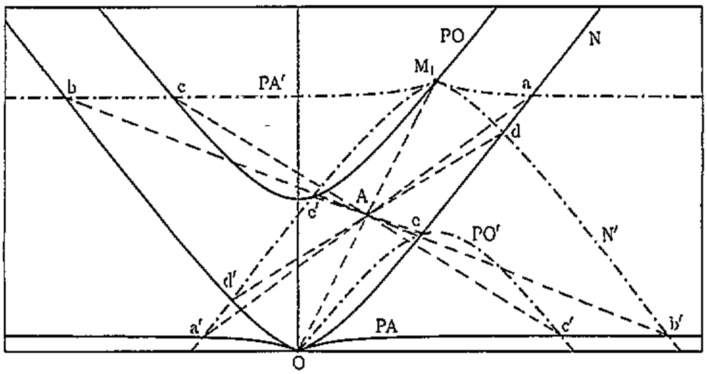


**Figure 2.** Construction solving the resonance condition (4), case $M_1 \in$ PO.

(ii) $M_1 \in$ N. Figure 3 gives the construction. As in the first case we have two solutions (2a), (2b) with $M_2$ on the PO branch, with opposite directions of propagation, and one (2c) with $M_2$ on the same branch N as $M_1$, but in the opposite direction; in the three cases $M_3$ is on the PA branch. No analogue to solutions (1d) and (1e) exists here. Solution (2c) exists for every value of $\omega_1$, but solutions (2a), (2b) exist only if $\omega_1$ is bigger than a certain value

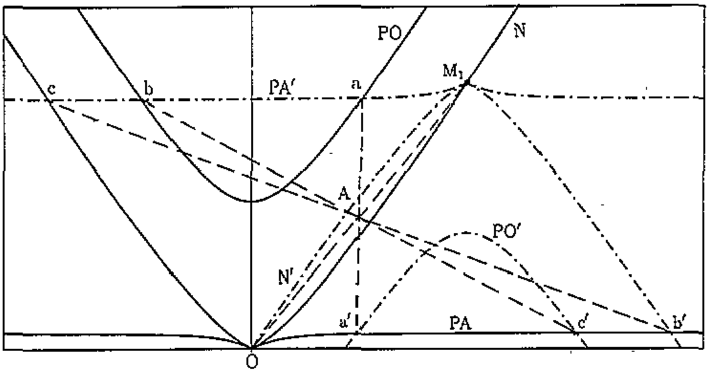


**Figure 3.** Construction solving the resonance condition (4), case $M_1 \in N$.

the order of magnitude of which is $\omega_l + \omega_0$ ($\omega_l$ is the limiting value of $\omega$ on the PA branch as $k \longrightarrow +\infty$, and $\omega_0$ the value of $\omega$ on the PO branch as $k = 0$).

(iii) $M_1 \in$ PA. It is easy to see that in this case there is no solution.

## 4. First and second classes of interaction: stimulated backscattering.

The interaction occurring here is stimulated backscattering (SBS). As an 'optical' wave encounters an 'acoustic' wave, with a small amount of power, the 'optical' wave is reflected, in a way analogous to stimulated Brillouin scattering. Some nonlinear properties of SBS have to be noted. First, a small amount of acoutic wave power stimulates the backscattering, but a large amount of it enhances the effect. Second, the acoustic wave increases during the interaction. This growth will soon stop the power transmission from the incident to the backscattered wave and, at least for long pulses, the amount of transmitted power is not very large. Furthermore, the amplitude of both acoustic and backscattered waves, after the interaction, shows a modulation oscillating in time and space. Note also that, when the solution of the 3WRI system has SBS-type behaviour, the so-called 'incident' and 'backscattered' waves do not necessarily have different directions in the laboratory frame. This is a consequence of the Galilean invariance of the 3WRI system.

We study first a set of solutions of the resonance condition for which the interaction is SBS-type, and also where the wave is backscattered in the usual sense. We divide this set into two classes, depending upon whether or not the backscattered waves belong to the same polarization mode as the incident wave. The first class (SBS with change of polarization) contains the following particular cases: (1b), where $M_1$ is on the PO branch, $M_2$ on the N branch, with a negative wavenumber $k_2$, and $M_3$ on the PA branch; and (2b), which is similar to the previous one, except that $M_1$ is on the N branch and $M_2$ on the PO one, the signs of $k_1$ and $k_2$ being unchanged. The second class of interaction (SBS without change of polarization) contains the two cases (1c), where $M_1$ and $M_2$ are both in the PO branch, in opposite directions, and $M_3$ on the PA branch, and (2c) which is similar to the latter except that $M_1$ and $M_2$ are both on the N branch.

We can choose $\omega_1 = \omega$ as a free parameter in this problem and, for large values of $\omega$, we can give explicit approximate expressions of all the quantities involved. The dispersion relation admits asymptotic developments so that, on branch PO, as $\omega \longrightarrow +\infty$:

$$k = \omega - \frac{m_x}{2} - \frac{q}{\omega} + \mathrm{O}\left(\frac{1}{\omega^2}\right) \tag{18}$$

with

$$q = \frac{m^2}{8}[4\alpha + 1 + (1 - 2\alpha)\sin^2\varphi] \tag{19}$$

where $m$, $\varphi$ are defined by

$$m_x = m\cos\varphi \qquad m_t = m\sin\varphi. \tag{20}$$

On the branch N, as $\omega \longrightarrow +\infty$:

$$k = \omega + \frac{m_x}{2} - \frac{q}{\omega} + \mathrm{O}\left(\frac{1}{\omega^2}\right). \tag{21}$$

On the branch PA, $\omega$ tends to

$$\omega_l = m\sqrt{\alpha(\alpha + \sin^2\varphi)} \tag{22}$$

as $k$ tends to infinity and we have

$$k \sim m\sqrt{\frac{2\alpha + (1-\alpha)\sin^2\varphi}{2(1 - \omega/\omega_l)}}. \tag{23}$$

To study the four cases (1b), (2b), (1c) and (2c) we immediately introduce two parameters $\epsilon = \pm 1$, $\eta = \pm 1$ and write the following system verified by the $(k_j, \omega_j)$:

$$\begin{aligned}
k_1 &= \omega - \epsilon\frac{m_x}{2} - \frac{q}{\omega} + \mathrm{O}\left(\frac{1}{\omega^2}\right) \\
-k_2 &= \omega_2 + \eta\epsilon\frac{m_x}{2} - \frac{q}{\omega_2} + \mathrm{O}\left(\frac{1}{\omega_2^2}\right) \\
k_3 &\sim m\sqrt{\frac{2\alpha + (1-\alpha)\sin^2\varphi}{2(1 - \omega_3/\omega_l)}} \\
\omega_1 &= \omega = \omega_2 + \omega_3 \\
k_1 &= k_2 + k_3.
\end{aligned} \tag{24}$$

$\eta = +1$ corresponds to the first class, and then, for $\epsilon = +1$, we get the case (1b) ($M_1$ on the PO branch) and for $\epsilon = -1$, the case (2b). $\eta = -1$ corresponds to the second class, and then, for $\epsilon = +1$, we get the case (1c) ($M_1$ and $M_2$ on the PO branch) and for $\epsilon = -1$, the case (2c).

By solving the system (24) at the limit $\omega \longrightarrow +\infty$, we obtain the following solutions.

(i) Using (11), the group velocities

$$\begin{aligned}
V_1 &= 1 + \mathrm{O}\left(\frac{1}{\omega^2}\right) \\
V_2 &= -1 + \mathrm{O}\left(\frac{1}{\omega^2}\right) \\
V_3 &\sim \frac{1}{8}\left(\frac{m}{\omega}\right)^3 (2\alpha + (1-\alpha)\sin^2\varphi)\sqrt{\alpha(\alpha + \sin^2\varphi)}
\end{aligned} \tag{25}$$

so that

$$V_2 < V_3 < V_1. \tag{26}$$

(ii) Using (12), (13) and (A16), the interaction constants are: for the first class ($\eta = 1$),

$$\begin{aligned} A_1 &= -4\frac{\omega^3}{m^2}\tan\varphi\frac{1}{(\alpha(\alpha+\sin^2\varphi))^2}K_\epsilon \\ A_2 &= -A_1 \\ A_3 &= \frac{m^7}{32}\frac{\sin\varphi\cos\varphi}{\alpha\omega^6}(\alpha(\alpha+\sin^2\varphi))^{3/2}K_\epsilon \end{aligned} \tag{27}$$

with

$$K_\epsilon = \alpha(\alpha+2-\sin^2\varphi)+\epsilon\cos\varphi(2+\alpha)\sqrt{\alpha(\alpha+\sin^2\varphi)} \tag{28}$$

and, for the second class ($\eta = -1$),

$$\begin{aligned} A_1 &= -8\epsilon\frac{\omega^4}{m^3}\frac{\sin\varphi}{\alpha(\alpha+\sin^2\varphi)^2} \\ A_2 &= -A_1 \\ A_3 &= \frac{\epsilon}{16}\frac{m^6}{\omega^5}(\alpha(\alpha+\sin^2\varphi))^{3/2}\cos^2\varphi\sin\varphi. \end{aligned} \tag{29}$$

We see that, in each case,

$$\delta_3 = \delta_2 = -\delta_1 \tag{30}$$

where $\delta_j = \mathrm{sgn}(A_j)$.

The two conditions (24) and (30) allow us to identify the type of 3WRI present. Here we are in the SBS case. The system does not admit any soliton solution, but is still integrable by the IST method. We refer the reader to the cited articles, particularly part V of [4], for this resolution. The principal effect is an energy transfer from the faster $(\omega_1, k_1)$ wave to the backscattered one $(\omega_2, k_2)$, stimulated by the 'acoustic' one $(\omega_3, k_3)$. In terms of the $q_j$ the transfer is almost complete if the amplitude of the third wave, although small relative to the first, is large enough. The fields are given in terms of the solution $q_1$, $q_2$, $q_3$ of system (15) by

$$\begin{aligned} \boldsymbol{H}_1^1 &= \hat{h}_1^1 q_1 \\ \boldsymbol{H}_j^1 &= \hat{h}_j^1 q_j^* \qquad \text{for } j = 2,3 \end{aligned} \tag{31a}$$

$$\begin{aligned} \boldsymbol{M}_1^1 &= \hat{m}_1^1 q_1 \\ \boldsymbol{M}_j^1 &= \hat{m}_j^1 q_j^* \qquad \text{for } j = 2,3 \end{aligned} \tag{31b}$$

with, for the first class,

$$\hat{h}_j^1 = -2\sqrt{2}\left(\frac{\omega}{m}\right)^{3/2}\frac{\alpha(1+\sin^2\varphi/\alpha)^{1/4}\cos\varphi}{|K_\epsilon\sin\varphi|}\begin{pmatrix}0\\ \mathrm{i}\\ (-1)^j\epsilon\end{pmatrix} \qquad \text{for } j = 1,2 \tag{32a}$$

$$\hat{h}_3^1 = 4\frac{\omega}{m}\frac{\alpha\sin\varphi|\cos\varphi|}{|K_\epsilon\sin\varphi|}\begin{pmatrix}\mathrm{i}\\ 0\\ 0\end{pmatrix} \tag{32b}$$

$$\hat{m}_j^1 = (-1)^{j+1}2\sqrt{2}\sqrt{\frac{\omega}{m}}\frac{\epsilon\alpha(1+\sin^2\varphi/\alpha)^{1/4}\cos\varphi}{|K_\epsilon\sin\varphi|}\begin{pmatrix}-\mathrm{i}\sin\varphi\\ \mathrm{i}\cos\varphi\\ (-1)^j\epsilon\cos\varphi\end{pmatrix} \qquad \text{for } j = 1,2 \tag{33a}$$

$$\hat{m}_3^1 = 4\frac{\omega}{m}\frac{\alpha|\cos\varphi|}{|K_\epsilon\sin\varphi|}\begin{pmatrix}-\mathrm{i}\sin\varphi\\ \mathrm{i}\cos\varphi\\ -\sqrt{1+\sin^2\varphi/\alpha}\end{pmatrix} \tag{33b}$$

where $K_\epsilon$ is given by (28). For the second class we have:

$$\hat{h}_1^1 = \hat{h}_2^1 = \sqrt{2}\sqrt{\frac{\omega}{m}}\frac{\cos\varphi}{|\sin\varphi\cos\varphi|}\left(1+\frac{\sin^2\varphi}{\alpha}\right)^{1/4}\begin{pmatrix}0\\-\mathrm{i}\\\epsilon\end{pmatrix} \tag{34a}$$

$$\hat{h}_3^1 = 2\frac{\sin\varphi}{|\sin\varphi|}\begin{pmatrix}\mathrm{i}\\0\\0\end{pmatrix} \tag{34b}$$

$$\hat{m}_1^1 = \hat{m}_2^1 = \epsilon\sqrt{2}\sqrt{\frac{m}{\omega}}\frac{\cos\varphi}{|\sin\varphi\cos\varphi|}\left(1+\frac{\sin^2\varphi}{\alpha}\right)^{1/4}\begin{pmatrix}-\mathrm{i}\sin\varphi\\\mathrm{i}\cos\varphi\\-\epsilon\cos\varphi\end{pmatrix} \tag{35a}$$

$$\hat{m}_3^1 = \frac{2}{|\sin\varphi|}\begin{pmatrix}-\mathrm{i}\sin\varphi\\\mathrm{i}\cos\varphi\\-\sqrt{1+\frac{\sin^2\varphi}{\alpha}}\end{pmatrix}. \tag{35b}$$

To answer the question whether the size of vector coefficients $\hat{h}_j^1$, $\hat{m}_j^1$ changes the transfer coefficients from wave 1 to wave 2 and so on, and thus the qualitative results obtained in solving system (15), we introduce an 'effective norm' $\|\cdot\|_{\text{eff}}$ such that the mean value, in a duration large in regard to the $t$ scale and small in regard to the $\tau$ scale, of the first wave $\boldsymbol{H}_1^1 e^{i\phi_1} + \boldsymbol{H}_1^{1*}e^{-i\phi_1}$ would be $\|\hat{h}_1^1\|_{\text{eff}}\cdot|q_1|$, and so on. Thus

$$\|\boldsymbol{u}\|_{\text{eff}} = \sqrt{2\boldsymbol{u}\cdot\boldsymbol{u}^*} \tag{36}$$

for every complex vector $\boldsymbol{u}$.

Let us call $R$ the factor due to these terms in the reflection coefficient, i.e.

$$R = \frac{\|\hat{h}_1^2\|_{\text{eff}}}{\|\hat{h}_1^1\|_{\text{eff}}}. \tag{37}$$

We see that, in every case, $R = 1$, so that the reflection coefficient calculated solving system (15) by the IST method is exactly the same as that which occurs here. Furthermore, we see that

$$Q = \frac{\|\hat{h}_1^3\|_{\text{eff}}}{\|\hat{h}_1^1\|_{\text{eff}}} = \sqrt{\frac{m}{\omega}}|\sin\varphi|\left(1+\frac{\sin^2\varphi}{\alpha}\right)^{-1/4} \tag{38}$$

which order of magnitude is the relatively small number $\sqrt{m/\omega}$. In regard to the amplitude of the first wave, the amplitudes of the 'acoustic' wave that occur in our problem are smaller than those given by the IST method resolution of (15) by this factor $Q$.

## 5. Third class of interactions: transition points between SBS and soliton exchange interaction

This case is much more complex than either of the two preceding ones. It regroups the following solutions of the resonance condition:

(i) (1a)(1d)(1e): where $M_1$ is on the PO branch, $M_2$ on the N one, both with positive $k$; $M_3$ tends to a finite limit as $\omega_1 = \omega$ tends to infinity, and may belong to every branch of the dispersion relation; we thus get the three subcases (1a), (1d), (1e).

(ii) (2a) is analoguous to the preceding one but $M_2$ is on PO and $M_1$ on N, and $M_3$ is always on the PA branch.

In order to deal with all these subcases at once, we must use new notation. Let us consider a PO wave $(k_P, \omega_P)$ with large positive $k_P$ and $\omega_P$, an N wave $(k_N, \omega_N)$, $k_N$ and

$\omega_N$ being also large and positive, and a third wave ($k_F$, $\omega_F$), with finite $k_F$ and $\omega_F$ (with no further assumption on them), such that

$$\omega_P = \omega_N + \omega_F \qquad k_P = k_N + k_F. \tag{39}$$

We will find the (1a), (1d), (1e) cases for $M_1 = M_P$, $M_2 = M_N$, $M_3 = M_F$ and the (2a) one for $M_1 = M_N$, $M_2 = M_P$, $M_3(-k_F, \omega_F)$.

As previously, we restrict ourselves to the case where the parameter $\omega = \omega_P$ is very large.

We use the asymptotic developments (18) for $k_P$ as a function of $\omega_P = \omega$, and (21) for $k_F$ as a function of $\omega_N = \omega - \omega_F$. At the limit $\omega \longrightarrow +\infty$, $k_F$ and $\omega_F$ verify the system formed by the dispersion relation (3) and the relation

$$k_F = \omega_F - m_x \tag{40}$$

obtained by reporting the expansions (18), (21) into (39).

Thus if we put $X = k_F/\omega_F = 1/u_F$, $X$ is a solution of the following fourth-degree equation:

$$[1+\alpha(1-X^2)]^2\cos^2\varphi + (1-X^2)[1+\alpha(1-X^2)](1+\alpha)\sin^2\varphi = (1+X)^2\cos^2\varphi. \tag{41}$$

The solution of this equation is, in the general case, very complicated, thus the whole discussion of the system's behaviour can only be achieved in a few particular cases.

First we see that we can restrict ourselves to $\varphi \in [0, \frac{1}{2}\pi]$. By an appropriate choice of the orientation of the $y$ axis, we can impose $\sin\varphi > 0$. A change of sign of $\cos\varphi$ corresponds to a change in the propagation direction. The P waves become the N ones and, conversely, in our equations but one can verify that no qualitative result is modified.

We have completely solved the problem in the following three particular cases:

(i) an exterior field parallel to the propagation direction: $\varphi = 0$;

(ii) an exterior field perpendicular to the propagation direction: $\varphi = \frac{1}{2}\pi$;

(iii) a strong exterior field: $\alpha \longrightarrow +\infty$.

However, in the second case the interaction coefficients vanish, and thus our perturbation theory is no longer valid. No further mention of this case will be made in this paper, and we will give the results for the remaining two cases, that have an interesting physical meaning.

The computation of the constants appearing in system (10) necessitates some technical work: appendix B is devoted to the determination of the relative magnitude of the group velocities and the sign of the interaction constants that characterizes the type of interaction. Some more detail on the expression of the interaction constants is given in appendix C. The calculus works as follows: first we solve equation (41), that solves the resonance condition (section B.1, equation (B2)). Then the group velocities of the three waves can be calculated: both $V_N$ and $V_P$ are close to 1 as $\omega \longrightarrow +\infty$, their relative magnitude can be determined as $\varphi = 0$, and also as $\alpha \longrightarrow +\infty$ (section B.2). We find that $V_N > V_P$ for the cases (2a), (1a), and (1d), and $V_N < V_P$ for the case (1e). $V_F$ is then computed (equations (B15) to (B20)): in every case $V_F < 1$, even at the limit, thus $V_F < V_N$, $V_P$. Next, we find the expressions for the interaction constants as functions of the parameter $X$, these are formulae (B1) to (B26) in the third section of appendix B. The signs of $A_P = -A_N$ and $A_F$ depend mainly on the sign of quantities $\mathcal{P}_P(X)$, $\mathcal{P}_F(X)$ defined by equations (B24), (B25). We replace $X$ by the solutions (B2), (B4) in these expressions and determine their sign depending on the values of $\alpha$ (for $\varphi = 0$) and $\varphi$ (as $\alpha \longrightarrow +\infty$) (equations (B27) to (B32)).

Summarizing all these results, we are able to discuss and describe the type of 3WRI involved depending on the cases and values of $\alpha$, $\varphi$.

In addition to the SBS interaction, another and very rich interaction type occurs, which is called the soliton exchange interaction. For a complete description of the involved phenomena, we refer the reader to [4] (part IV), of which we only recall here the main results. The most important effect is that any soliton contained in a wave with an intermediate velocity disappears, giving rise to two solitons, one in each wave. The word 'contained' has to be understood in terms of the number of normal modes in the Zakharov–Shabat (ZS) scattering data of the wave; i.e. the characteristic profile of this soliton does not need to appear in the initial wave but the daughter waves have the typical aspect of $N$-soliton solutions. This transfer appears as soon as the slightest amount of one of the two waves with extreme velocities is present in the interaction region; random noise is sufficient. A transfer of the radiation energy of the wave of intermediate velocity to one of the others is also possible. It occurs when the third wave has a large amplitude. Transfer of solitons from the waves with extreme velocities to one with intermediate velocity is also possible, if the initial state contains one (or more) pair of resonant solitons (one in each wave). Afterwards the resultant soliton disappears, giving rise to the two initial solitons again. This interaction seems to be ineffective but is not: arbitrarily shaped pulses which in terms of the ZS scattering data contain solitons that after the interaction take the shape of the usual $N$-soliton solution.

Our results are as follows.

*Case (1a).* We have here $V_F < V_P < V_N$. In the $\varphi = 0$ case, if $\alpha < \alpha_1$, $\alpha_1 \simeq 0.465$, $\delta_P = -\delta_N = \delta_F$ thus the interaction is a soliton exchange; each soliton of the branch N gives rise to one soliton in the branch PA and one in the branch PO. If $\alpha > \alpha_1$, $\delta_P = -\delta_N = \delta_F$. Thus we have SBS of the wave P of PO type into the wave F of PA type ($V_F < 0$). For large values of $\alpha$, the type of interaction depends on $\varphi$: if $0 \leqslant \varphi < \varphi_0$, where $\varphi_0 = \frac{1}{2}\arccos(-\frac{1}{3})$, $\delta_P = -\delta_N = \delta_F$, thus this is the SBS case; if $\varphi_0 < \varphi \leqslant \frac{1}{2}\pi$, $\delta_P = -\delta_N = -\delta_F$ it is soliton exchange.

*Case (1d).* Here $V_F < V_P < V_N$. If $\alpha$ is large, we have $\delta_P = -\delta_N = \delta_F$, and this describes SBS of wave N to wave F (both of N type) ($V_F < 0$). If $\varphi = 0$, we have, in contrast, $\delta_F = \delta_N = -\delta_P$ , so that we have a soliton exchange interaction between wave P and both others. These two results may seem contradictory, but in fact the leading term of expression (B31*a*) *for* $\mathcal{P}_P$ as $\alpha \longrightarrow +\infty$ *vanishes at* $\varphi = 0$, *thus this expression is no longer valid for* $\varphi = 0$.

*Case (1e).* In this case $V_F < V_N < V_P$. For large values of $\alpha$ we do not have any transition between the two interaction types: we have for every value of $\varphi$: $\delta_P = -\delta_N = -\delta_F$ and SBS of the P wave to the F one. Note that the 'backscattered' wave has the same direction as the incident one. Putting $\varphi = 0$, we have the same interaction type: SBS, except in a particular window: if $\alpha_2 < \alpha < 1+\sqrt{\frac{11}{3}}$, where $\alpha_2 \simeq 1.119$, then $\delta_P = -\delta_N = \delta_F$ and there is a soliton exchange between the P wave and both others.

*Case (2a).* This is the last and simplest case: $V_F < V_P < V_N$ and $\delta_P = -\delta_N = -\delta_F$, and thus there is a soliton exchange interaction between the wave P and both others.

## 6. Conclusion and perspectives

We have studied the interaction of three waves in a ferromagnetic dielectric: first we have shown that it is governed by the integrable 3WRI system, then we have classified the cases where a three-wave interaction occurs and, for high frequencies, we have performed a complete characterization of the solutions. For two classes of solution we have shown that the interaction is of SBS type and, for the third one, we saw that, as the strength of the exterior magnetic field or the angle between it and the propagation direction varies, transitions between two regimes, one where the interaction is of SBS type and one where it is soliton exchange, occur.

We will pursue the nonlinear study of electromagnetic waves in ferromagnetic dielectrics, studying the non-resonant interaction of two waves; this should lead to a nonlinear Faraday effect. We also study wave modulation in $2+1$ dimensions and, taking account of damping, the nonlinear propagation of a signal given at the origin of space.

## Acknowledgments

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## Appendix A. Derivation of the 3WRI system

In this appendix we want to show how we derive the 3WRI system (10) from the fundamental equations (1), (2). After rescaling it by

$$\boldsymbol{H} \longrightarrow \frac{\mu_0 \delta \boldsymbol{H}}{c} \qquad \boldsymbol{M} \longrightarrow \frac{\mu_0 \delta \boldsymbol{M}}{c} \qquad t \longrightarrow ct \tag{A1}$$

and assuming that $\boldsymbol{M}$, $\boldsymbol{H}$ depend on $t$ and $x$ only, (1), (2) become

$$\frac{\partial^2}{\partial t^2}(H^x + M^x) = 0 \tag{A2$a$}$$

$$\frac{\partial^2}{\partial t^2}(H^s + M^s) = \frac{\partial^2}{\partial x^2} H^s \qquad \text{for } s = y, z \tag{A2$b$}$$

$$\frac{\partial}{\partial t}\boldsymbol{M} = -\boldsymbol{M} \wedge \boldsymbol{H}. \tag{A3}$$

We introduce the phases $\phi_j$ (5), the development of $\boldsymbol{H}$ (6), and an analogue development of $\boldsymbol{M}$, and assume the resonance condition (4).

For $j = \pm 1, \pm 2, \pm 3$, in the sum $\sum_{n \in \mathbb{Z}^3} \boldsymbol{H}_n^2 \mathrm{e}^{n \cdot \phi}$ there are many terms proportional to $\mathrm{e}^{\mathrm{i}\phi_j}$. Only the sum of all these terms can be defined in a unique way: we call it $\boldsymbol{H}_j^2 \mathrm{e}^{\mathrm{i}\phi_j}$; $\boldsymbol{H}_j^2$ is a function of the stretched variables $\xi$, $\tau$. Let $A$ be the set of all the indices $n = (n_1, n_2, n_3)$ such that $\mathrm{e}^{\mathrm{i}n \cdot \phi} \neq \mathrm{e}^{\mathrm{i}\phi_j}$ for all $j = \pm 1, \pm 2$ or $\pm 3$.

We get the following expansion of $\boldsymbol{H}$:

$$\boldsymbol{H} = \boldsymbol{H}^0 + \varepsilon \sum_j \boldsymbol{H}_j^1 \mathrm{e}^{\mathrm{i}\phi_j} + \varepsilon^2 \sum_j \boldsymbol{M}_j^2 \mathrm{e}^{\mathrm{i}\phi_j} + \varepsilon^2 \sum_{n \in A} \boldsymbol{H}_n^2 \mathrm{e}^{\mathrm{i}n \cdot \phi} + \mathrm{O}(\varepsilon^3) \tag{A4}$$

and an analogous one for $\boldsymbol{M}$.

We have the reality condition $\boldsymbol{M}_{-j}^1 = \boldsymbol{M}_j^{1*}$, and so on. We insert these expansions into equations (A2$a$), (A2$b$), (A3), and collect terms of the same order in $\varepsilon$.

At order $\varepsilon^0$, assuming that $\boldsymbol{H}^0$, $\boldsymbol{M}^0$ are constants, we find only one condition: $\boldsymbol{M}^0 \wedge \boldsymbol{H}^0 = \boldsymbol{0}$, so we may define $\alpha$ such that $\boldsymbol{H}^0 = \alpha \boldsymbol{M}^0 = \alpha \boldsymbol{m}$.

At order $\varepsilon^1$, we get the following equations, for each $j$:

$$M_j^{1,x} + H_j^{1,x} = 0 \tag{A5$a$}$$

$$M_j^{1,s} + H_j^{1,s} = \left(\frac{k_j}{\omega_j}\right)^2 H_j^{1,s} \qquad \text{for } s = y, z \tag{A5$b$}$$

$$\mathrm{i}\omega_j \boldsymbol{M}_j^1 = \boldsymbol{m} \wedge [\boldsymbol{H}_j^1 - \alpha \boldsymbol{M}_j^1]. \tag{A6}$$

For each value of $j = 1, 2, 3$, these equations constitute a linear homogeneous system for the components of $\boldsymbol{H}_j^1$, $\boldsymbol{M}_j^1$. Each has non-trivial solutions only if the following condition (dispersion relation) holds:

$$\mu_j^2 m_x^2 + \gamma_j \mu_j (1+\alpha) m_t^2 = \gamma_j^2 \omega_j^2 \tag{A7}$$

where we put

$$\begin{aligned} \gamma_j &= 1 - \left(\frac{k_j}{\omega_j}\right)^2 \\ \mu_j &= 1 + \alpha\gamma_j. \end{aligned} \tag{A8}$$

The solutions have the form (8) where

$$\begin{aligned} m_j^{1,x} &= -h_j^{1,x} = -\mathrm{i}\gamma_j \mu_j m_t \\ m_j^{1,y} &= -\gamma_j h_j^{1,y} = +\mathrm{i}\gamma_j \mu_j m_x \\ m_j^{1,z} &= -\gamma_j h_j^{1,z} = -\gamma_j^2 \omega. \end{aligned} \tag{A9}$$

At order $\varepsilon^2$, we make no attempt to determine the terms in $\mathrm{e}^{\mathrm{i}n\cdot\phi}$, $n \in A$; we only need to know that they are allowed to exist. Regarding the term proportional to $\mathrm{e}^{\mathrm{i}\phi_j}$, we obtain the equations:

$$\frac{2\mathrm{i}}{\omega_j}\frac{\partial}{\partial\tau}(M_j^{1,x} + H_j^{1,x}) + M_j^{2,x} + H_j^{2,x} = 0 \tag{A10$a$}$$

$$\frac{2\mathrm{i}}{\omega_j}\frac{\partial}{\partial\tau}(M_j^{1,s} + H_j^{1,s}) + M_j^{2,s} + H_j^{2,s} = -\frac{2\mathrm{i}k_j}{\omega_j^2}\frac{\partial}{\partial\xi}H_j^{1,s} + \frac{k_j^2}{\omega_j^2}H_j^{2,s} \qquad \text{for } s = y, z \tag{A10$b$}$$

$$\frac{\partial}{\partial\tau}\boldsymbol{M}_j^1 - i\omega_j \boldsymbol{M}_j^2 = -\sum_{(p,r)/\phi_p+\phi_r=\phi_j} \boldsymbol{M}_p^1 \wedge \boldsymbol{H}_r^1 - \boldsymbol{m} \wedge (\boldsymbol{H}_j^2 - \alpha\boldsymbol{M}_j^2). \tag{A11}$$

From (A10) we get, using (A9),

$$M_j^{2,x} = -H_j^{2,x} \tag{A12$a$}$$

$$M_j^{2,s} = -\gamma_j H_j^{2,s} - \frac{2i}{\omega_j^2}\left[k_j\frac{\partial}{\partial\xi} + (1-\gamma_j)\omega_j\frac{\partial}{\partial\tau}\right]H_j^{1,s} \qquad \text{for } s = y, z. \tag{A12$b$}$$

Inserting these expressions into (A11), we obtain a linear $3 \times 3$ system for $H_j^{2,x}$, $H_j^{2,y}$, $H_j^{2,z}$ for each $j$; the determinant of this system is proportional to

$$\mu_j^2 m_x^2 + \gamma_j \mu_j (1+\alpha) m_t^2 - \gamma_j^2 \omega_j^2$$

and is thus zero, according to the dispersion relation. We replace one column (the third one) of the matrix of the system by the right-hand side of the latter, and compute the corresponding determinant. This gives the condition for the existence of solutions. Multiplying this condition by the constants:

$$\frac{-\mu_j u_j^2}{2\Gamma_j\omega_j}\frac{1}{(b_j + 1 + \gamma_j\mu_j u_j^2)} \tag{A13}$$

we obtain system (10).

In this calculus one must pay attention to the term coming from

$$\boldsymbol{B}_j = \sum_{(p,r)/\phi_p+\phi_r=\phi_j} \boldsymbol{M}_p^1 \wedge \boldsymbol{H}_r^1.$$

We have

$$B_1 = M_2^1 \wedge H_3^1 + M_3^1 \wedge H_2^1 \tag{A14}$$

and the corresponding term in system (9) is $A_1 g_2 g_3$, with $A_1$ given by (11), (12).

$B_2$, $B_3$ are slightly different:

$$B_2 = M_1^1 \wedge H_3^{1*} + M_3^{1*} \wedge H_1^1 \tag{A15}$$

giving rise to a term $A_2 g_1 g_3^*$, involving the complex conjugate of $g_3$, and (still using (11)) there are some sign changes in the expression of $A_2'$ in regard to $A_1'$:

$$\begin{aligned} A_2' = \omega_2 m_x m_t\{&-\gamma_2\omega_2\mu_1\mu_3[\gamma_1(1-\gamma_3)+\gamma_3(1-\gamma_1)] \\ &+\mu_2\gamma_1\gamma_3[\mu_1\omega_3(1-\gamma_3)-\mu_3\omega_1(1-\gamma_1)] \\ &-\gamma_2(1+\alpha)(\gamma_1-\gamma_3)[\mu_3\gamma_1\omega_1+\mu_1\gamma_3\omega_3]\}. \end{aligned} \tag{A16}$$

$B_3$ is wholly analogous to $B_2$, and gives rise to the term $A_3 g_1 g_2^*$. The expression of $A_3'$ is deduced from (A16) by permuting indices 2 and 3.

## Appendix B. Determination of the type of interaction for the third class

In this appendix we give the asymptotic solutions of the resonance condition for the third class of interactions, and the corresponding calculus of velocities and interaction constants.

### *Appendix B.1. Solutions of equation (41)*

*General case.* A graphic solution is given in figure B1. We have plotted the dispersion relation $\omega(k)$ and drawn the straight line $\Delta$ of equation $\omega = k + m_x$. The asymptotes of the branches PO and N of the dispersion relation are drawn with dotted lines; they have the equations $\omega = \pm k \pm \frac{1}{2}m_x 2$. We obtain four solutions; each of which corresponds to one of the cases (2a), (1a), (1d), (1e) as given in figure B1.

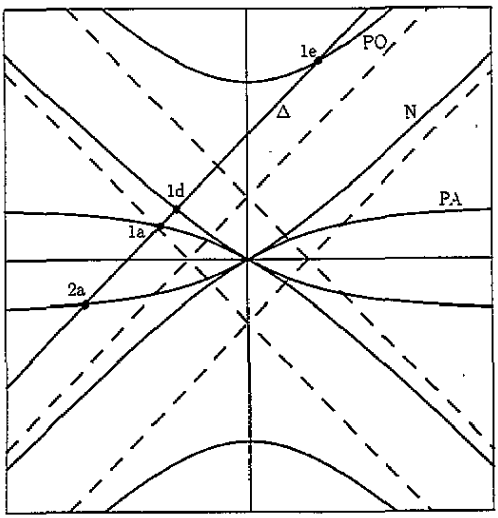


**Figure B1.** Construction solving equation (46).

Because $X = k_F/\omega_F$, we get from the graphic the following inequalities:

$$X_{1a} < X_{1d} < -1 < 0 < X_{1e} < 1 < X_{2a} \tag{B1}$$

where $X_{1a}$ is the solution $X$ that corresponds to the case (1a), and so on.

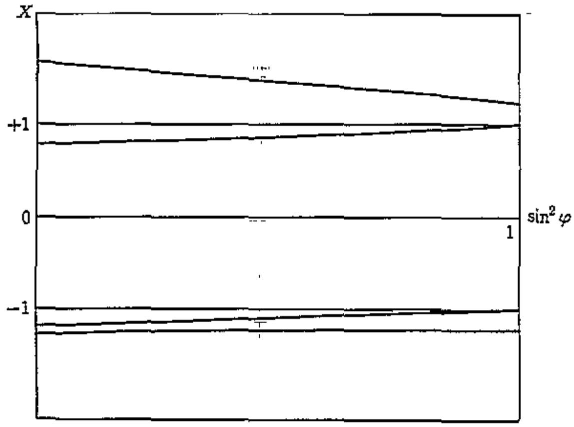


**Figure B2.** Plot of $X = k_F/\omega_F$ as a function of $\sin^2\varphi$ for a given $\alpha$ ($\alpha = 2$).

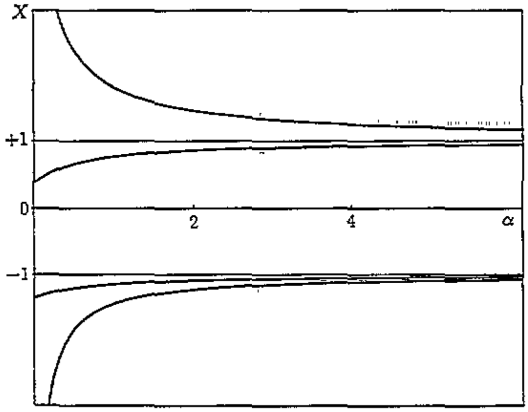


**Figure B3.** Plot of $X = k_F/\omega_F$ as a function of $\alpha$ for a given $\varphi$ ($\varphi = \frac{1}{4}\pi$).

A numerical solution may be obtained; we give two plots of $X$, in figure B2 as a function of $\sin^2\varphi$ for a given $\alpha$, in figure B3 as a function of $\alpha$ for a given value of $\varphi$. This numerical approach shows that the variation in $X$ as a function of $\alpha$ and $\varphi$ is weak, thus we may hope that the description of the two particular cases $\varphi = 0$ and $\alpha \longrightarrow +\infty$ give a reasonable approximate description of the general case. Furthermore we can see that the approximate solution we give below is a really good one, e.g. for $\alpha = 2$ and $\varphi = \frac{1}{4}\pi$, we get $X_{2a} \simeq 1.445$ from the formula (B4*d*) and $X_{2a} \simeq 1.464$ numerically. For $\alpha = 2$, we get from formula (B4*a*) $X_{1a} \simeq 1.25$, and the numerically calculated $X_{1a}$ varies from $-1.225$ for $\varphi = \frac{1}{2}\pi$ to $-1.28$ for $\varphi = 0$.

*Case* $\varphi = 0$. We obtain the solutions:

$$X_{\varepsilon,\eta} = \frac{\varepsilon - \eta\sqrt{1 + 4\alpha(1+\varepsilon) + 4\alpha^2}}{2\alpha} \qquad \varepsilon, \eta = \pm 1 \tag{B2}$$

with

$$X_{1a} = X_{-+} \qquad X_{1d} = X_{++} \qquad X_{1e} = X_{--} \qquad X_{2a} = X_{+-}. \tag{B3}$$

*Case* $\alpha \longrightarrow +\infty$. We obtain:

$$X_{1a} = -1 - \frac{1}{2\alpha} + O\left(\frac{1}{\alpha^2}\right) \tag{B4a}$$

$$X_{1d} = -1 - \frac{\cos^2\varphi}{2\alpha} + O\left(\frac{1}{\alpha^2}\right) \tag{B4b}$$

$$X_{1e} = 1 + \frac{1}{4\alpha}\left(1 + \cos^2\varphi - \sqrt{16\cos^2\varphi + \sin^4\varphi}\right) + O\left(\frac{1}{\alpha^2}\right) \tag{B4c}$$

$$X_{2a} = 1 + \frac{1}{4\alpha}\left(1 + \cos^2\varphi + \sqrt{16\cos^2\varphi + \sin^4\varphi}\right) + O\left(\frac{1}{\alpha^2}\right). \tag{B4d}$$

In the calculus of the velocity $V_F$ an extra term is necessary for the (1a) case:

$$X_{1a} = -1 - \frac{1}{2\alpha} + \frac{1}{8\alpha^2} + O\left(\frac{1}{\alpha^3}\right). \tag{B5}$$

*Appendix B.2. Calculus of velocities*

*Appendix B.2.1. Waves with large values of* $\omega$. We use the following asymptotic development of $k$:

$$k = \omega - \epsilon\frac{m\cos\varphi}{2} - \frac{q}{\omega} - \epsilon\frac{w}{\omega^2} + O\left(\frac{1}{\omega^3}\right) \tag{B6}$$

where $\epsilon = +1$ for the PO wave, and $-1$ for the N wave, $q$ is given by (19) and $w$ by

$$w = \frac{m^3}{128\cos\varphi}[8 + 28\alpha + 35\alpha^2 + 4\alpha(2 + 7\alpha)\cos 2\varphi + \alpha(\alpha - 4)\cos 4\varphi] \tag{B7}$$

using (B5) in (11) we obtain

$$V = 1 - \frac{q}{\omega^2} - \epsilon\frac{2w}{\omega^3} + O\left(\frac{1}{\omega^4}\right). \tag{B8}$$

Now we calculate $V_P$, the velocity of a PO wave ($\epsilon = +1$) with $\omega_P = \omega \gg 1$, and $V_N$, the velocity of an N wave ($\epsilon = -1$) with $\omega_N = \omega - \omega_F$; we obtain

$$V_N - V_P = \frac{-2[q\omega_F + 2w]}{\omega^3} + \left(\frac{1}{\omega^4}\right). \tag{B9}$$

Thus

$$V_N > V_P \qquad \text{when } \omega_F < \omega_{Fl} \tag{B10a}$$

where

$$\omega_{Fl} = -2w/q. \tag{B10b}$$

*Case* $\varphi = 0$. We obtain

$$\omega_{Fl} = m\frac{1 + 4\alpha + 8\alpha^2}{1 + 4\alpha} \tag{B11}$$

$\omega_F$ can be expressed as a function of $X$ as follows.

$$\omega_F = \frac{m\cos\varphi}{1 - X} \tag{B12}$$

and inserting the solution (B2) into (B12), we write the condition (B10) and obtain the following result: for the cases (2a), (1a) and (1d), $V_N > V_P$, and for the case (1e) $V_N < V_P$.

*Case* $\alpha \longrightarrow +\infty$. We obtain

$$\omega_{Fl} \sim \frac{m\alpha}{8} \frac{35 + 28\cos 2\varphi + \cos 4\varphi}{\cos\varphi(3 + \cos 2\varphi)} \tag{B13}$$

inserting the solution (B4) into (B12), we can explicit the condition (B10) and thus we get $V_N > V_P$ for the cases (1a), (1d), (2a) and $V_N < V_P$ for the case (1e). Note that the result is the same in both cases.

*Appendix B.2.2. Wave* $(k_F, \omega_F)$. Using (11), we obtain the expression for the third wave velocity $V_F$ as a function of the solution $X$:

$$V_F = \frac{X[(1 + \alpha(1 - X^2))^2 + (1 + X)^2]}{X^2[(1 + \alpha(1 - X^2))^2 + (1 + X)^2] + (1 + X)^2(1 - X^2)(1 + \alpha(1 - X^2))}. \tag{B14}$$

*Case* $\varphi = 0$. The solution $X$ is given by (B2) for $\varepsilon = +1$: we obtain

$$V_F = -\left(\frac{2 + 15\alpha + 8\alpha^2 + \eta(2 + 7\alpha + 4\alpha^2)\sqrt{1 + 8\alpha + 4\alpha^2}}{5 + 42\alpha + 28\alpha^2 + 4\alpha^3}\right) \tag{B15}$$

$\eta = +1$ corresponds to case (1d), $\eta = -1$ to case (2a).

For $\varepsilon = +1$ we obtain

$$V_F = \frac{\alpha[1 - \eta(1 + 2\alpha)\sqrt{1 + 4\alpha^2}]}{1 + 8\alpha + 4\alpha^2 + 4\alpha^3} \tag{B16}$$

$\eta = +1$ corresponds to case (1a), $\eta = -1$ to case (1e).

*Case* $\alpha \longrightarrow +\infty$. Using the asymptotic values (B4)–(B5) of $X$ in (B14) we obtain: in the (1a) case

$$V_F = -1 + \frac{1}{2\alpha} + O\left(\frac{1}{\alpha^2}\right) \tag{B17}$$

in the (1d) case

$$V_F = -1 + \frac{\cos^2\varphi}{2\alpha} + O\left(\frac{1}{\alpha^2}\right) \tag{B18}$$

in the two other cases

$$V_F = 1 - \frac{\lambda_\varepsilon}{\alpha} \frac{(14 + 2\cos^2\varphi)\lambda_\varepsilon - 3\sin^2\varphi}{5 + 3\cos^2\varphi - 2\lambda_\varepsilon \sin^2\varphi} + O\left(\frac{1}{\alpha^2}\right) \tag{B19}$$

where

$$\lambda_\varepsilon = \frac{1}{4}\left(1 + \cos^2\varphi + \varepsilon\sqrt{16\cos^2\varphi + \sin^4\varphi}\right) \tag{B20}$$

is such that $X = 1 + \lambda_\varepsilon/\alpha + O(1/\alpha^2)$. The (2a) case is obtained for $\varepsilon = +1$, the (1e) one for $\varepsilon = -1$.

*Appendix B.3. Computation of the* $A_1$, $A_2$, $A_3$ *coefficients*

*Appendix B.3.1. General properties.* We use the expansions (18) and (21) of $k_P$ and $k_N$ in the expressions (13), (A16) of $A_N$, $A_P$, $A_F$ (these expressions are correct if we put $(P, N, F) \equiv (1, 2, 3)$ because they assume that $\omega_1 = \omega_2 + \omega_3$, $k_1 = k_2 + k_3$ and nothing on the signs of the $\omega_j$ and $k_j$). We obtain first that

$$A_N = -A_P \tag{B21}$$

at leading order in $\omega$. Then we get the expressions of $A_{\mathrm{N}}$ and $A_{\mathrm{P}}$ as functions of $X$. They read as follows.

$$A_{\mathrm{P}} = -\frac{m^2}{8\omega}\frac{\sin\varphi}{\cos\varphi}\mathcal{P}_{\mathrm{P}}(X) + \mathrm{O}\left(\frac{1}{\omega^2}\right) \tag{B22}$$

$$A_{\mathrm{F}} = -\frac{m^3}{4\omega^2}\sin\varphi\frac{1+\alpha(1-X^2)}{1-X}\frac{\mathcal{P}_{\mathrm{F}}(X)}{\mathcal{D}_{\mathrm{F}}(X)} + \mathrm{O}\left(\frac{1}{\omega^3}\right) \tag{B23}$$

where

$$\begin{aligned}\mathcal{P}_{\mathrm{P}}(X) = {} & (1-X^2)[1+\alpha(1-X^2)][2\alpha+(1-\alpha)\sin^2\varphi] \\ & +2\cos^2\varphi([1+\alpha(1-X^2)][(2-\alpha)(1-X^2)-1] \\ & +(1+X)[(2+\alpha)(1-X^2)-1])\end{aligned} \tag{B24}$$

$$\begin{aligned}\mathcal{P}_{\mathrm{F}}(X) = {} & -(1-X^2)[2\alpha+(1-\alpha)\sin^2\varphi][1+2(1+\alpha)(1-X)] \\ & +2\cos^2\varphi(2(1-X)(1+\alpha)[\alpha(1-X^2)-1]+(\alpha-2)(1-X^2)+2+X)\end{aligned} \tag{B25}$$

$$\mathcal{D}_{\mathrm{F}}(X) = X^2([1+\alpha(1-X^2)]^2+(1+X)^2)+(1+X)^2(1-X^2)[1+\alpha(1-X^2)]. \tag{B26}$$

In the two particular cases $\varphi = 0$ and $\alpha \longrightarrow +\infty$, we can give more explicit expressions for these terms and determine their signs.

*Appendix B.3.2. Case $\varphi = 0$.* If $\varepsilon = 1$, we obtain the expressions:

$$\mathcal{P}_{\mathrm{P}}(X) = \frac{1}{\alpha^2}\left[-1-7\alpha-6\alpha^2+\eta(1+3\alpha)\sqrt{1+8\alpha+4\alpha^2}\right] \tag{B27a}$$

$$\mathcal{P}_{\mathrm{F}}(X) = \frac{1}{\alpha^2}\left[2+11\alpha+2\alpha^2-4\alpha^3-\eta(2+3\alpha+2\alpha^2)\sqrt{1+8\alpha+4\alpha^2}\right] \tag{B27b}$$

(The reader is referred to (B3) to know to which of the cases (1a), (1d), (1e) and (2a) corresponds each value of $(\varepsilon, \eta)$.) We see that $\mathcal{P}_{\mathrm{P}}(X)$ has the sign of $\eta$, and $\mathcal{P}_{\mathrm{F}}(X)$ the opposite sign.

If $\varepsilon = -1$, we get the expressions:

$$\mathcal{P}_{\mathrm{P}}(X) = \frac{1}{\alpha^3}\left[4-3\alpha+9\alpha^2-2\alpha^3+\eta(4-3\alpha+\alpha^2)\sqrt{1+4\alpha^2}\right] \tag{B28a}$$

$$\mathcal{P}_{\mathrm{P}}(X) = \frac{1}{\alpha^2}\left[2-3\alpha-2\alpha^2-4\alpha^3+\eta(2-3\alpha-2\alpha^2)\sqrt{1+4\alpha^2}\right]. \tag{B28b}$$

For $\eta = +1$, $\mathcal{P}_{\mathrm{P}}(X)$ is always positive, but for $\eta = -1$, its sign changes depending on $\alpha$: $\mathcal{P}_{\mathrm{P}}(X) > 0$ if and only if $\alpha < 1+\sqrt{\frac{11}{3}}$. The sign of $\mathcal{P}_{\mathrm{F}}(X)$ is not constant either: it depends on the position of $\alpha$ in relation to the two values $\alpha_1$, $\alpha_2$ defined as the two positive roots of the equation:

$$-4+8\alpha+5\alpha^2-8\alpha^3 = 0 \tag{B29}$$

with $\alpha_1 < \alpha_2$. We have $\alpha_1 \simeq 0.465$ and $\alpha_2 \simeq 1.119$.

For $\eta = +1$, $\mathcal{P}_{\mathrm{F}} > 0$ if and only if $\alpha < \alpha_1$.

For $\eta = -1$, $\mathcal{P}_{\mathrm{F}} > 0$ if and only if $\alpha > \alpha_2$.

*Appendix B.3.3. Case $\alpha \longrightarrow +\infty$ Case (1a).* We obtain the expressions:

$$\mathcal{P}_{\mathrm{P}}(X) = \frac{2\cos^2\varphi}{\alpha} + \mathrm{O}\left(\frac{1}{\alpha^2}\right) \tag{B30a}$$

$$\mathcal{P}_{\mathrm{F}}(X) = -2(1+3\cos 2\varphi)\alpha + \mathrm{O}(1). \tag{B30b}$$

$\mathcal{P}_{\mathrm{P}}(X)$ is always positive, but the sign of $\mathcal{P}_{\mathrm{F}}(X)$ depends on $\varphi$. Let $\varphi_0 = \frac{1}{2}\arccos(-\frac{1}{3})$, then $\mathcal{P}_{\mathrm{F}}(X) > 0$ if and only if $\varphi < \varphi_0$ (for $\varphi \in [0, \frac{1}{2}\pi[$).

*Case (1d).* We obtain the expressions:

$$\mathcal{P}_{\mathrm{P}}(X) = \tfrac{1}{8}(-5+\cos 2\varphi)\sin^2 2\varphi + \mathrm{O}\left(\frac{1}{\alpha}\right) \tag{B31a}$$

$$\mathcal{P}_{\mathrm{F}}(X) = -2\cos^2\varphi(3+\cos 2\varphi)\alpha + \mathrm{O}(1). \tag{B31b}$$

Both $\mathcal{P}_{\mathrm{P}}(X)$ and $\mathcal{P}_{\mathrm{F}}(X)$ are always negative.

*Cases (1e)–(2a).* We obtain the expressions:

$$\mathcal{P}_{\mathrm{P}}(X) = -\mathcal{P}_{\mathrm{F}}(X) = -\frac{\cos^2\varphi}{16}(3+\cos 2\varphi) \times \left[30 + 2\cos 2\varphi + \varepsilon\sqrt{2(67+60\cos 2\varphi + \cos 4\varphi)}\right] \tag{B32}$$

where $\varepsilon = +1$ in the case (2a) and $\varepsilon = -1$ in the case (1e). Here we have always $\mathcal{P}_{\mathrm{P}}(X) < 0$, and thus $\mathcal{P}_{\mathrm{F}}(X) > 0$.

$\mathcal{D}_{\mathrm{F}}$ is always positive; in fact, $V_{\mathrm{F}} = X[(1+\alpha(1-X^2)) + (1+X)^2]/\mathcal{D}_{\mathrm{F}}$, so that $\mathcal{D}_{\mathrm{F}}$ has the same sign as $V_{\mathrm{F}}X = V_{\mathrm{F}}/u_{\mathrm{F}}$, and this quotient of group and phase velocities cannot be negative. Other factors are $(1-X)$ with known sign, and $[1+\alpha(1-X^2)] = \mu_{\mathrm{F}}$ which is negative only when $\mathrm{M_F}$ is on the branch PA.

*Summarizing the data in this appendix, we obtain the results described in section 5.*

## Appendix C. Interaction coefficients in the cases (1a) (1d) (1e) and (2a)

$A_{\mathrm{P}}$ is given by (B22), where $\mathcal{P}_{\mathrm{P}}(X)$, given by (B24), has the expressions (B27a), (B28*a*), (B30*a*), (B31*a*), (B32) depending on the case. Expressions for $A_{\mathrm{F}}$ are more complicated.

First we rewrite (B25) as

$$A_{\mathrm{F}} = \frac{-m^3}{4\omega^2}\sin\varphi \frac{\mu_{\mathrm{F}}}{1-X}\frac{\mathcal{P}_{\mathrm{F}}(X)}{\mathcal{D}_{\mathrm{F}}(X)} \tag{C1}$$

where $\mu_{\mathrm{F}} = 1+\alpha(1-X^2)$.

We will give expressions for $\mu_{\mathrm{F}}/(1-X)$ and $\mathcal{D}_{\mathrm{F}}(X)$ in every studied case ($\mathcal{P}_{\mathrm{F}}(X)$ has the expressions (B27*b*), (B28*b*), (B30*b*), (B31*b*), (B32) depending on the case) without attempting to reduce further the expression of $A_{\mathrm{F}}$ itself.

*Case $\varphi = 0$.* If $\varepsilon = +1$,

$$\mathcal{D}_{\mathrm{F}}(X) = \frac{1}{2\alpha^5}\left[1 + 15\alpha + 71\alpha^2 + 116\alpha^3 + 60\alpha^4 + 8\alpha^5 - \eta(1+11\alpha+33\alpha^2+26\alpha^3+4\alpha^4)\sqrt{1+8\alpha+4\alpha^2}\right]. \tag{C2a}$$

If $\varepsilon = -1$,

$$\mathcal{D}_{\mathrm{F}}(X) = \frac{1}{2\alpha^5}\left[1 - \alpha + 3\alpha^2 - 4\alpha^4 + 8\alpha^5 + \eta(1-\alpha+\alpha^2+2\alpha^3-4\alpha^4)\sqrt{1+4\alpha^2}\right]. \tag{C2b}$$

In both cases,

$$\frac{\mu_{\mathrm{F}}}{1-X}=\frac{4\alpha+2(1+\varepsilon)-\eta\sqrt{1+4(1+\varepsilon)\alpha+4\alpha^2}}{2(\varepsilon+2)}. \tag{C3}$$

We recall the values of $(\varepsilon,\eta)$ corresponding to each case:

$$(1a)\text{: } (-,+) \qquad (1d)\text{: } (+,+) \qquad (1e)\text{: } (-,-) \qquad (2a)\text{: } (+,-).$$

*Case* $\alpha \longrightarrow +\infty$. For case (1a), we need a development of $X$ to the third order in $\alpha$:

$$X_{1a}=-1-\frac{1}{2\alpha}+\frac{1}{8\alpha^2}-\frac{2\cos^2\varphi+1}{16\sin^2\varphi}\frac{1}{\alpha^3}+\mathrm{O}\left(\frac{1}{\alpha^4}\right). \tag{C4}$$

We obtain

$$\mathcal{D}_{\mathrm{F}}(X)=\frac{1}{4\alpha^2}+\frac{1}{8\alpha^3}+\mathrm{O}\left(\frac{1}{\alpha^4}\right) \tag{C5}$$

$$\frac{\mu_{\mathrm{F}}}{1-X}=\frac{-\cos^2\varphi}{8\alpha^2\sin^2\varphi}+\mathrm{O}\left(\frac{1}{\alpha^3}\right). \tag{C6}$$

For case (1d), we obtain

$$\mathcal{D}_{\mathrm{F}}(X)=\sin^4\varphi+\mathrm{O}\left(\frac{1}{\alpha}\right) \tag{C7}$$

$$\frac{\mu_{\mathrm{F}}}{1-X}=\frac{\sin^2\varphi}{2}+\mathrm{O}\left(\frac{1}{\alpha}\right). \tag{C8}$$

For the cases (2a) ($\epsilon=+1$) and (1e) ($\epsilon=-1$), we obtain

$$\mathcal{D}_{\mathrm{F}}(X)=\frac{1}{4}\left[17+14\cos^2\varphi+\cos^4\varphi+\sin^4\varphi-\epsilon(2-\cos^2\varphi)\sqrt{16\cos^2\varphi+\sin^4\varphi}\right] \tag{C9}$$

$$\frac{\mu_{\mathrm{F}}}{1-X}=\frac{-\sin^2\varphi+\epsilon\sqrt{16\cos^2\varphi+\sin^4\varphi}}{2\left(1+\cos^2\varphi+\epsilon\sqrt{16\cos^2\varphi+\sin^4\varphi}\right)}. \tag{C10}$$

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